# Message passing in TrueSkill

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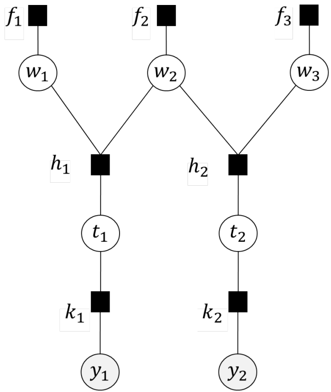
# Key concepts

- we attempt to apply message passing to TrueSkill
- we encounter two problems
  - the TrueSkill graph isn't a tree
    - we will ignore this problem, but message passing becomes *iterative*
  - some of the messages don't have standard form
    - approximate using moment matching (seperate chunk)
- we write out messages in excruciating detail

# The full TrueSkill graph




Prior factors: $f_i(w_i) = \mathcal{N}(w_i; \mu_0, \sigma_0^2)$

"Game" factors:

$$h_g(w_{I_g}, w_{J_g}, t_g) = \mathcal{N}(t_g; w_{I_g} - w_{J_g}, 1)$$

($I_g$ and $J_g$ are the players in game g)

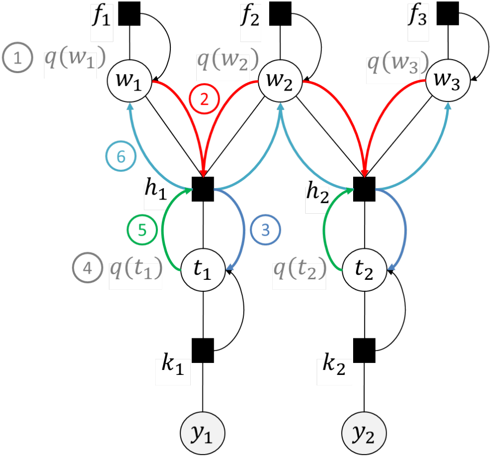
Outcome factors:

$$k_g(t_g, y_g) = \delta\big(y_g - \text{sign}(t_g)\big)$$

We are interested in the marginal distributions of the skills $w_i$.

- What shape do these distributions have?
- We need to make some approximations.
- We will also pretend the structure is a tree (ignore loops).

# Expectation Propagation in the full TrueSkill graph




Iterate

(1) Update skill marginals.

(2) Compute skill to game messages.

(3) Compute game to performance messages.

(4) Approximate performance marginals.

(5) Compute performance to game messages.

(6) Compute game to skill messages.

# Message passing for TrueSkill

$$
\begin{aligned}
m^{\tau=0}_{h_g\to w_{I_g}}(w_{I_g}) &= 1, \quad m^{\tau=0}_{h_g\to w_{J_g}}(w_{J_g}) = 1, \quad \forall\, g,\\
q^{\tau}(w_i) &= f(w_i)\prod_{g=1}^{N} m^{\tau}_{h_g\to w_i}(w_i) \sim \mathcal{N}(\mu_i,\sigma_i^2),\\
m^{\tau}_{w_{I_g}\to h_g}(w_{I_g}) &= \frac{q^{\tau}(w_{I_g})}{m^{\tau}_{h_g\to w_{I_g}}(w_{I_g})}, \quad m^{\tau}_{w_{J_g}\to h_g}(w_{J_g}) = \frac{q^{\tau}(w_{J_g})}{m^{\tau}_{h_g\to w_{J_g}}(w_{J_g})},\\
m^{\tau}_{h_g\to t_g}(t_g) &= \iint h_g(t_g,w_{I_g},w_{J_g})\, m^{\tau}_{w_{I_g}\to h_g}(w_{I_g})\, m^{\tau}_{w_{J_g}\to h_g}(w_{J_g})\, dw_{I_g}\, dw_{J_g},\\
q^{\tau+1}(t_g) &= \mathrm{Approx}\big(m^{\tau}_{h_g\to t_g}(t_g)\, m_{k_g\to t_g}(t_g)\big),\\
m^{\tau+1}_{t_g\to h_g}(t_g) &= \frac{q^{\tau+1}(t_g)}{m^{\tau}_{h_g\to t_g}(t_g)},\\
m^{\tau+1}_{h_g\to w_{I_g}}(w_{I_g}) &= \iint h_g(t_g,w_{I_g},w_{J_g})\, m^{\tau+1}_{t_g\to h_g}(t_g)\, m^{\tau}_{w_{J_g}\to h_g}(w_{J_g})\, dt_g\, dw_{J_g},\\
m^{\tau+1}_{h_g\to w_{J_g}}(w_{J_g}) &= \iint h_g(t_g,w_{I_g},w_{J_g})\, m^{\tau+1}_{t_g\to h_g}(t_g)\, m^{\tau}_{w_{I_g}\to h_g}(w_{I_g})\, dt_g\, dw_{I_g}.
\end{aligned}
$$

# In a little more detail

At iteration $\tau$ messages $m$ and marginals $q$ are Gaussian, with *means* $\mu$, *standard deviations* $\sigma$, *variances* $v = \sigma^2$, *precisions* $r = v^{-1}$ and *natural means* $\lambda = r\mu$.

**Step 0** Initialise incoming skill messages:

$$\left.\begin{aligned} r^{\tau=0}_{h_g \to w_i} &= 0 \\ \mu^{\tau=0}_{h_g \to w_i} &= 0 \end{aligned}\right\} m^{\tau=0}_{h_g \to w_i}(w_i)$$

**Step 1** Compute marginal skills:

$$\left.\begin{aligned} r^{\tau}_{i} &= r_0 + \sum_g r^{\tau}_{h_g \to w_i} \\ \lambda^{\tau}_{i} &= \lambda_0 + \sum_g \lambda^{\tau}_{h_g \to w_i} \end{aligned}\right\} q^{\tau}(w_i)$$

**Step 2** Compute skill to game messages:

$$\left.\begin{aligned} r^{\tau}_{w_i \to h_g} &= r^{\tau}_{i} - r^{\tau}_{h_g \to w_i} \\ \lambda^{\tau}_{w_i \to h_g} &= \lambda^{\tau}_{i} - \lambda^{\tau}_{h_g \to w_i} \end{aligned}\right\} m^{\tau}_{w_i \to h_g}(w_i)$$

**Step 3** Game to performance messages:

$$\left.\begin{aligned} v^{\tau}_{h_g \to t_g} &= 1 + v^{\tau}_{w_{I_g} \to h_g} + v^{\tau}_{w_{J_g} \to h_g} \\ \mu^{\tau}_{h_g \to t_g} &= \mu^{\tau}_{I_g \to h_g} - \mu^{\tau}_{J_g \to h_g} \end{aligned}\right\} m^{\tau}_{h_g \to t_g}(t_g)$$

**Step 4** Compute marginal performances:

$$\begin{aligned} p(t_g) &\propto \mathcal{N}(\mu^{\tau}_{h_g \to t_g}, v^{\tau}_{h_g \to t_g})\mathbb{I}\big(y - \mathrm{sign}(t)\big) \\ &\simeq \mathcal{N}(\tilde{\mu}^{\tau+1}_g, \tilde{v}^{\tau+1}_g) \;=\; q^{\tau+1}(t_g) \end{aligned}$$

We find the parameters of $q$ by *moment matching*

$$\left.\begin{aligned} \tilde{v}^{\tau+1}_g &= v^{\tau}_{h_g \to t_g}\Big(1 - \Lambda\big(\tfrac{\mu^{\tau}_{h_g \to t_g}}{\sigma^{\tau}_{h_g \to t_g}}\big)\Big) \\ \tilde{\mu}^{\tau+1}_g &= \mu^{\tau}_{h_g \to t_g} + \sigma^{\tau}_{h_g \to t_g}\Psi\big(\tfrac{\mu^{\tau}_{h_g \to t_g}}{\sigma^{\tau}_{h_g \to t_g}}\big) \end{aligned}\right\} q^{\tau+1}(t_g)$$

where we have defined $\Psi(x) = \mathcal{N}(x)/\Phi(x)$ and $\Lambda(x) = \Psi(x)(\Psi(x) + x)$.

**Step 5** Performance to game message:

$$\left.\begin{aligned} r^{\tau+1}_{t_g \to h_g} &= \tilde{r}^{\tau+1}_g - r^{\tau}_{h_g \to t_g} \\ \lambda^{\tau+1}_{t_g \to h_g} &= \tilde{\lambda}^{\tau+1}_g - \lambda^{\tau}_{h_g \to t_g} \end{aligned}\right\} m^{\tau+1}_{t_g \to h_g}(t_g)$$

**Step 6** Game to skill message:
For player 1 (the winner):

$$\left.\begin{aligned} \nu^{\tau+1}_{h_g \to w_{I_g}} &= 1 + \nu^{\tau+1}_{t_g \to h_g} + \nu^{\tau}_{w_{J_g} \to h_g} \\ \mu^{\tau+1}_{h_g \to w_{I_g}} &= \mu^{\tau}_{w_{J_g} \to h_g} + \mu^{\tau+1}_{t_g \to h_g} \end{aligned}\right\} m^{\tau+1}_{h_g \to w_{I_g}}(w_{I_g})$$

and for player 2 (the looser):

$$\left.\begin{aligned} \nu^{\tau+1}_{h_g \to w_{J_g}} &= 1 + \nu^{\tau+1}_{t_g \to h_g} + \nu^{\tau}_{w_{I_g} \to h_g} \\ \mu^{\tau+1}_{h_g \to w_{J_g}} &= \mu^{\tau}_{w_{I_g} \to h_g} - \mu^{\tau+1}_{t_g \to h_g} \end{aligned}\right\} m^{\tau+1}_{h_g \to w_{J_g}}(w_{J_g})$$

Go back to **Step 1** with $\tau := \tau + 1$ (or stop).